A Survey of Kleinian Groups: Infinite Covolume

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Fuchsian Groups

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$= PSL_2(\mathbb{R}) = Isom(\mathbb{H}^2)$
Metric = $ds^2 = \frac{dx^2+dy^2}{y^2}$ on upper half plane $\mathbb{H}^2$.
Metric blows up as on approaches $y = 0$ (resp. $z = 0$).
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Kleinian Groups: 3 Perspectives

Discrete subgroup G of group of Mobius transformations $Mob(\widehat{\mathbb{C}})$ Complex Analysis/Dynamics
Discrete subgroup G of $PSL_2(\mathbb{C})$ – Lie group theoretic.
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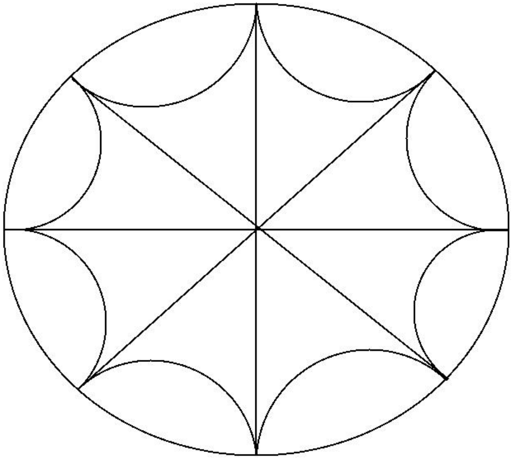

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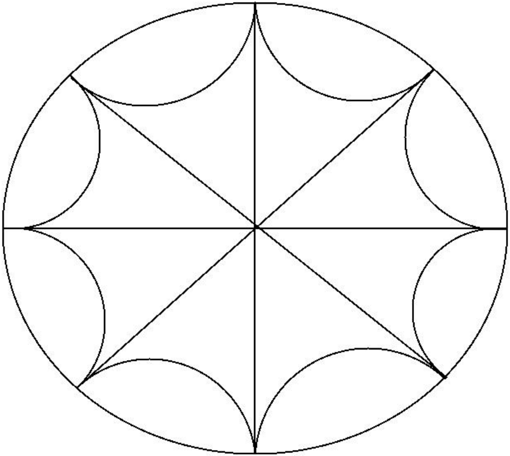

One way to generalize Fuchsian groups:

Fuchsian Group as an example of a Kleinian group

Limit set $\Lambda_G =$ Set of accumulation points in $\widehat{\mathbb{C}}$ of $G.o$ for some (any) $o \in \mathbb{H}^3$.

Hence for a Fuchsian group of the kind described above, limit set = round equatorial circle.

Example of an infinite covolume Kleinian group.

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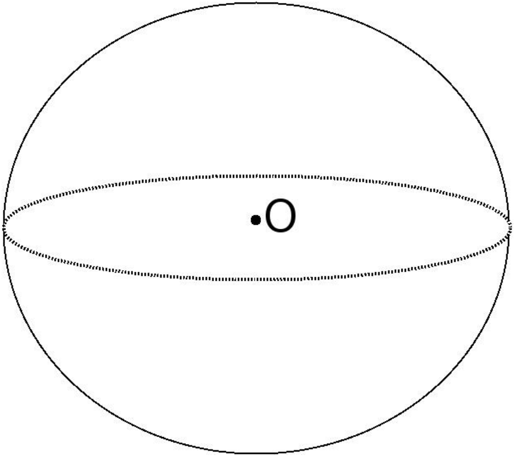
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Complement: Two round open discs.
On each, G acts freely (i.e. without fixed points) properly discontinuously, by conformal automorphisms.
Hence quotient is two copies of the 'same' Riemann surface (one dimensional complex analytic manifold.)
$\widehat{\mathbb{C}} \setminus \Lambda_G = \Omega_G$ is called the *domain of discontinuity* of G.

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Next set of examples of Kleinian groups come from trying to put different conformal structures on the two complementary pieces of the domain of discontinuity.

IMPORTANT NOTE: Conformal Structure on a 2 manifold is EQUIVALENT TO

Constant curvature metric (for us curvature = -1)

which is EQUIVALENT TO structure as a non-singular algebraic curve.

Poincare-Koebe-Klein Uniformization Theorem.

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Ahlfors-Bers simultaneous Uniformization Theorem:
Given any two conformal structures τ_1, τ_2 on a closed topological 2-manifold, there is a discrete subgroup G of $Mob(\widehat{\mathbb{C}})$ whose limit set is *topologically* a circle, and whose domain of discontinuity quotients to two Riemann surfaces τ_1, τ_2.
Limit set is the image under a quasiconformal map of the round circle.
These *(quasi Fuchsian)* groups can be thought of as *deformations* of Fuchsian groups (Lie group theoretically) or quasiconformal deformations (analytically).

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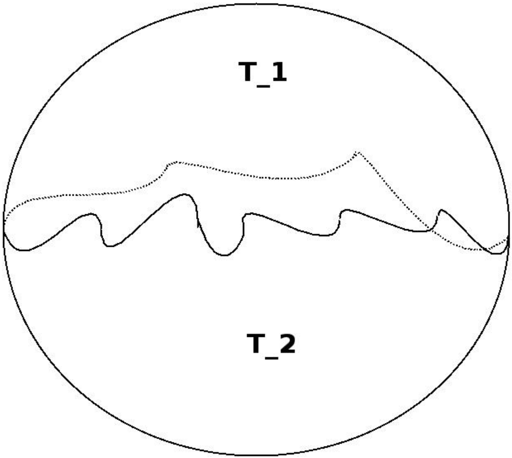
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Complexity of quasi Fuchsian group measured in terms of Hausdorff dimension.

How about geometric picture of these groups?
Convex hull CH_G of limit set Λ_G = smallest closed convex subset of $\mathbb{H}^3$ invariant under G.
Can be constructed by joining all pairs of points on limit set by bi-infinite geodesics and iterating this construction.
Quotient of CH_G by G is homeomorphic to $S \times I$, where $\pi_1(S)$ is isomorphic to G.
Called *Convex core* $CC(M)$ of $M = \mathbb{H}^3/G$.
Thickness (= 'length' of the I direction) of CH_G/G is a geometric measure of the complexity of the group G.

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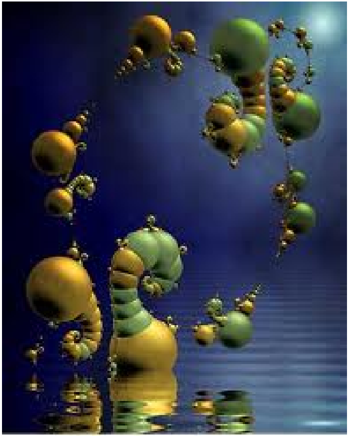
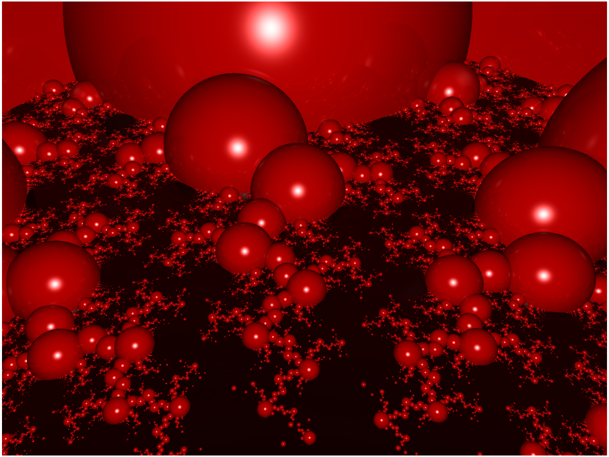


"Indra Family," by Jos Leys



Limits of quasiFuchsian groups:
Thickness of Convex core $CC(M)$ tends to infinity.
2 possibilities: Degenerate only τ_1. Degenerate both τ_1, τ_2.
i.e. $I \to [0, \infty)$ **(simply degenerate)**
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Lipman Bers:

- Examples exist.
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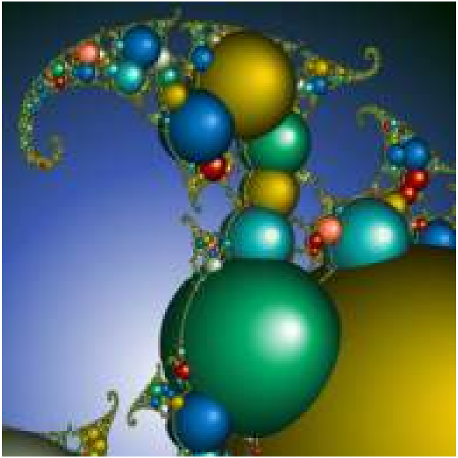
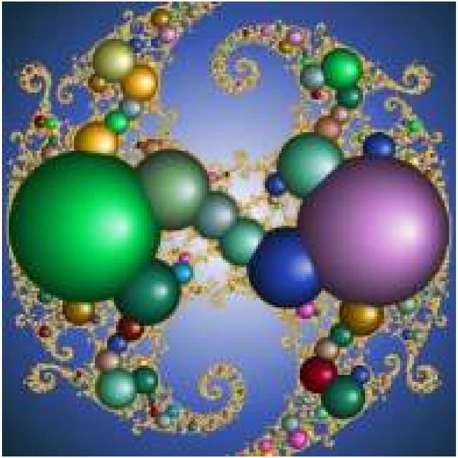
- **Question (Thurston): What does limit set go to?** In doubly degenerate case limit set of limiting group is all of $\widehat{\mathbb{C}}$.
- **Problem (Thurston)**: Understand these limits.





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1) **Ending Lamination Conjecture** (Proved by Minsky, Brock-Canary-Minsky): Ending laminations (pair of these in the doubly degenerate case) along with conformal structure on quotient of domain of discontinuity (in the simply degenerate case) is a complete invariant of the isometry type of M.

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Totally Degenerate Surface Groups

Consequences:

- Connected limit sets of f.g. (3d) Kleinian groups are locally connected
- There exist continuous boundary extensions: If Γ is the Cayley graph of a f.g. Kleinian group G, then (fixing a base point $0 \in \mathbb{H}^3$) the natural map $i : \Gamma \rightarrow \mathbb{H}^3$ extends continuously to a map $\hat{i} : \widehat{\Gamma} \rightarrow \widehat{\mathbb{H}^3}$ between the compactifications.
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Question: (Shalom) If Γ is a Zariski dense, infinite covolume, discrete subgroup of a semi-simple Lie group L, is Comm(Γ) discrete?

Answer: (M–) Yes, if

a) The limit set $\Lambda_\Gamma \subset \partial_F G$ (=Furstenberg boundary) is not invariant under a simple factor, OR

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