Riemann surfaces and their automorphism groups
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Riemann surfaces provide plenty of interesting groups and problems asociated with them: these include finite groups, since most compact Riemann surfaces have finite automorphism groups, and also finitely presented infinite groups, arising from the uniformisation of Riemann surfaces by Fuchsian groups. In many cases, these problems can be approached by using computational techniques.

Background reading. For an elementary introduction to Riemann surfaces and their automorphisms,:

G. A. Jones and D. Singerman, *Complex Functions*, Cambridge University Press.

At a rather more advanced level, and very comprehensive:

H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer.

For background on the combinatorial group theory aspects of Fuchsian and related groups:

H. Zieschang, E. Vogt and H-D. Coldewey, *Surfaces and Planar Discontinuous Groups*, Springer LNM 835, 1980.

For useful techniques for studying automorphism groups, though some of the results have now been overtaken:

R. D. M. Accola, *Topics in the Theory of Riemann Surfaces*, Springer LNM 1595, 1994.

For applications of character theory to Riemann surface automorphism groups:

T. Breuer, Characters and Automorphism Groups of Compact Riemann Surfaces, LMS Lecture Notes 280, 2000.

1 Riemann surfaces

1.1 Definitions

Informally, a *Riemann surface* X is a connected Hausdorff space which looks locally like a subset of the complex plane. More precisely we require that every point $p \in X$ has an open neighbourhood U with a homeomorphism $\phi : U \to V$ between U and an open subset $V \subseteq \mathbb{C}$. We call ϕ a *chart*, and the set of all such ϕ an *atlas* of charts. If we have charts $\phi : U \to V$ and $\phi' : U' \to V'$, with $U \cap U' \neq \emptyset$, then on $U \cap U'$ we have a coordinate transition function $\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U'), z \mapsto z'$; we require that these function should be analytic (differentiable everywhere). Thus the charts give complex local coordinates on X, allowing us to do analysis (differentiation, integration, etc); conformality of the coordinate transition functions means that the choice of charts around a point doesn't significantly affect the analysis we do.

Two atlases of charts on X are *compatible* if the coordinate transition functions from each to the other are analytic. This is an equivalence relation on atlases, and a *complex structure* on X is an equivalence class of atlases on X. A *Riemann surface* is a pair consisting of a connected Hausdorff space X and a complex structure on X. In general, a given surface X may have many (in fact uncountably many) different complex structures.

Every Riemann surface X is orientable: indeed, the positive orientation of $\mathbb{C}$ is transfered, via the charts, to an orientation of X. There is a similar but rather more general theory of *Klein surfaces*, which can be non-orientable and with boundary, but I will restrict my attention to Riemann surfaces. Fortunately, I will rarely need to use all the above formalism, and for most purposes the 'definition' in the first sentence will be adequate.

1.2 Examples

1. We can take X to be any open subset of $\mathbb{C}$ (including $\mathbb{C}$ itself), with a single chart consisting of the identity map. Two important examples are the upper half plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$$

and the unit disc

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 0\}.$$

2. Let X be the unit 2-sphere $S^2 \subset \mathbb{E}^3$. Identify the equatorial plane $x_3 = 0$ in $\mathbb{E}^3$ with $\mathbb{C}$ by identifying $(x_1, x_2, 0)$ with $x_1 + ix_2$. Stereographic projection from the north pole $n = (0, 0, 1)$ gives a homeomorphism ϕ from $U = S^2 \setminus \{n\}$ to $V = \mathbb{C}$, i.e. a chart; note that the south pole $s = (0, 0, -1)$ is sent to 0. A second chart $\phi' : U' = S^2 \setminus \{s\} \to V' = \mathbb{C}$ sends n to 0, and each other point $p \neq s$ to $1/\phi(p)$. On $U \cap U' = S^2 \setminus \{n, s\}$ the change of coordinates map $z \mapsto 1/z$ is conformal, since $z \neq 0$. Thus S^2 is a Riemann surface, called the *Riemann sphere* Σ. It is convenient to use ϕ to identify $S^2 \setminus \{n\}$ with $\mathbb{C}$, and to assign the coordinate ∞ to n, so that

$$\Sigma = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C}),$$

the complex projective line.

3. Let $X = \mathbb{C}/\Lambda$, where Λ is a *lattice* in $\mathbb{C}$, an additive subgroup generated by two elements which are linear independent over $\mathbb{R}$. Thus X is a topological group, with topology and group structure inherited from $\mathbb{C}$. Given any $p \in X$, a small enough disc U around p lifts to a disjoint union of discs in $\mathbb{C}$, each mapped homeomorphically onto U by the natural projection $\pi : \mathbb{C} \to X$. Choosing one of these as V we get a chart $\phi = \pi^{-1} : U \to V$. The change of coordinate maps are translations (by elements of Λ), so they are conformal. A Riemann surface of this type is called a *torus*.

4. As undergraduates, we learn how to construct the Riemann surface of a many-valued function, such as $\sqrt{z}$ or $\log z$, by joining copies of $\mathbb{C}$ across cuts. If we define the charts carefully, then the result is a Riemann surface as defined above, compact if and only if we start with an algebraic function. For instance, the Riemann surface of $\sqrt{(z-a)(z-b)(z-c)}$, with cuts between a and b, and between c and ∞, is a torus.

1.3 Automorphisms

An *isomorphism* $X \to Y$ of Riemann surface is a bijection which transforms local coordinates analytically. An *automorphism* of a Riemann surface X is an isomorphism $X \to X$; these form a group $\operatorname{Aut} X$.

1. The automorphism group of Σ is

$$\operatorname{Aut}\Sigma = PGL(2, \mathbb{C}),$$

consisting of the Möbius transformations

$$f : z \mapsto \frac{az+b}{cz+d} \quad (a, b, c, d \in \mathbb{C},\ ad - bc \neq 0),$$

where we define $f(\infty) = a/c$, and $f(z) = \infty$ if $cz + d = 0$. These are composed like 2×2 matrices, and scalar matrices λI ($\lambda \neq 0$) induce the identity automorphism, so $PGL(2, \mathbb{C}) \cong GL(2, \mathbb{C})/\{\lambda I\}$. Dividing the coefficients by $\sqrt{ad - bc}$, we may assume that $ad - bc = 1$, so $\operatorname{Aut}\Sigma = PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I\}$.

This group is sharply 3-transitive on Σ, meaning that, given two ordered triples of distinct elements of Σ, there is a unique automorphism taking one to the other, in the correct order. Firstly, if z_1, z_2, z_3 are distinct elements of $\mathbb{C}$ then

$$f(z) = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

is an automorphism sending z_1, z_2, z_3 to $0, 1, \infty$, and if some $z_i = \infty$ we take the limit of f as $z_i \to \infty$. This implies 3-transitivity, Sharpness follows from the fact that only the identity automorphism fixes three points (just solve the three simultaneous equations).

2. The automorphism group of $\mathbb{C}$ is the subgroup of $\mathrm{Aut}\,\Sigma$ fixing ∞, i.e. the group

$$\mathrm{Aut}\,\mathbb{C} = \mathrm{AGL}(1, \mathbb{C})$$

of affine transformations

$$f : z \mapsto az + b \quad (a, b \in \mathbb{C},\ a \neq 0)$$

of $\mathbb{C}$. This group is sharply 2-transitive on $\mathbb{C}$.

3. A *circle* C in Σ is the intersection of $\Sigma = S^2$ with a plane $\Pi \subset \mathbb{E}^3$ such that $|\Sigma \cap \Pi| > 1$. It corresponds to a Euclidean circle in $\mathbb{C}$, or a set of the form $\hat{L} = L \cup \{\infty\}$ where L is a Euclidean line in $\mathbb{C}$. A *disc* in Σ is one of the two open sets bounded by a circle. There is a unique circle through any three distinct points, so the 3-transitivity of $\mathrm{Aut}\,\Sigma$ implies that it is transitive on circles; it easily follows that it is transitive on discs, e.g. $z \mapsto 1/z$ transposes the two discs $\mathbb{H}$ and $\overline{\mathbb{H}}$ (the lower half plane) bounded by the circle $\hat{\mathbb{R}}$. Thus all discs are isomorphic as Riemann surfaces, and in particular $\mathbb{H} \cong \mathbb{D}$; in fact, the automorphism

$$f : z \mapsto \frac{z - i}{-iz + 1}$$

of Σ, representing a quarter-turn of Σ about the axis through ± 1, transforms $\mathbb{H}$ to $\mathbb{D}$. It follows that the automorphism groups of all discs are isomorphic. The automorphism group of $\mathbb{H}$ is

$$\mathrm{Aut}\,\mathbb{H} = PSL(2, \mathbb{R}),$$

the group of Möbius transformations

$$f : z \mapsto \frac{az + b}{cz + d} \quad (a, b, c, d \in \mathbb{R},\ ad - bc = 1).$$

(Here one could write $ad - bc > 0$, but dividing coefficients by $\sqrt{ad - bc}$ allows one to assume that $ad - bc = 1$.) This group acts transitively on $\mathbb{H}$, and 2-transitively on the boundary circle $\partial\mathbb{H} = \hat{\mathbb{R}}$. It is a subgroup of index 2 in $PGL(2, \mathbb{R})$, which is sharply 3-transitive on $\partial\mathbb{H}$; here elements with $ad - bc < 0$ transpose $\mathbb{H}$ and $\overline{\mathbb{H}}$ (see the example $z \mapsto 1/z$ above).

2 Uniformisation

2.1 The fundamental group

The fundamental group $\pi_1 T$ of a path-connected topopological space T measures how many holes T has. Two continuous paths $\gamma, \gamma' : I = [0, 1] \to T$, with the same end-points a and b, are *homotopic* if each can be continuously deformed into the other, entirely within T, keeping the end-points fixed. This is an equivalence relation. If we take $a = b$, so that the paths are all closed, then composing paths, and reversing the direction of a path induce multiplication and inversion on these equivalence classes; these equivalence classes then form a group, the *fundamental group* $\pi_1(T, a)$ of T based at a, with the identity element representing the constant path $\gamma(t) = a$ for all $t \in I$. Up to isomorphism, this group is independent of a, so we often write simply $\pi_1 T$. We say that T is *simply connected* if $\pi_1 T$ is trivial, i.e. every closed path in T can be continuously deformed to a point.

Example 1. The spaces $\mathbb{C}, \mathbb{H}, \mathbb{D}$ and Σ are all simply connected.

Example 2. The space $T = \mathbb{C} \setminus \{0\}$ has $\pi_1 T \cong \mathbf{Z}$: two closed paths in T are homotopic if and only if they go the same number $n \in \mathbf{Z}$ of times around the origin, where going around in the clockwise direction counts as negative. More generally, if we make r holes in $\mathbb{C}$ by removing r points or discs, the resulting fundmanetal group is a free group of rank r generated by the equivalence classes of the paths going once around a single hole.

Example 3. The fundamental group $\pi_1 T$ of a torus T is isomorphic to $\mathbb{Z}^2$: think of an element (m, n) as representing how many time a closed path winds around T in two different directions. More precisely, form T by identifying opposite sides of a parallelogram F, and take a base-point formed from the four corners of F (which become a single point in T). A closed path γ going once around the boundary of F is represented as a commutator $a^{-1}b^{-1}ab$, where a and b represent closed paths α and β following two successive sides, so that a^{-1} and b^{-1} represent the reverse paths. This path γ can be continuously deformed to a point within F, so it represents the identity element. Thus $[a, b] = 1$, that is, a and b commute. Now a and b generate $\pi_1 T$, since any closed path is equivalent to a combination of copies of α and β, and they have infinite order, so $\pi_1 T \cong \mathbb{Z}^2$.

Example 4. A compact, connected, orientable surface T is homeomorphic to a sphere with a finite number $g \geq 0$ of handles attached; we call g the *genus* of T. One can show that the fundamental group has a presentation

$$\pi_1 T = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle,$$

where a_i and b_i represent closed paths going in different directions around the ith handle. In particular, this applies to every compact Riemann surface. The basic idea is similar to Example 3 (which is just the case $g = 1$), except that now we use a $4g$-gon with sides identified in pairs to get the defining relation.

2.2 The Uniformisation Theorem

A group Γ acting by homeomorphisms on a topological space T acts *discontinuously* if every $p \in T$ has an open neighbourhood U such that $U \cap g(U) = \emptyset$ for all non-identity $g \in \Gamma$; this implies that the action is fixed-point-free, i.e. only the identity element has fixed points. Covering space theory tells us that every path-connected topological space T has a simply connected *universal covering space* $\tilde{T}$, and $T \cong \tilde{T}/\Gamma$ where Γ is a discontinuous group acting on $\tilde{T}$, isomorphic to the fundamental group $\pi_1 T$. (Summarising the construction, $\tilde{T}$

is the set of homotopy classes of paths from a chosen base point to an arbitrary point in T, the fundamental group $\pi_1 T$ acts on this space by composition of paths, and the projection $\tilde{T} \to T$ is given by forgetting the paths but remembering where they end.)

This all applies if T is a Riemann surface X, in which case the universal covering space $\tilde{X}$ has a natural complex structure, lifted from that of X, so that $\tilde{X}$ is a Riemann surface, and Γ acts as a group of automorphisms of $\tilde{X}$. We then say that X is *uniformised* by Γ.

Example. If X is a torus $\mathbb{C}/\Lambda$ then $\tilde{X} = \mathbb{C}$ and $\Gamma = \Lambda$, acting on $\mathbb{C}$ by translations and isomorphic to $\pi_1 X \cong \mathbb{Z}^2$.

This means that one can study all Riemann surfaces by studying the simply connected Riemann surfaces and their automorphism groups. By a miracle, known as the Poincaré-Koebe Uniformsiation Theorem, there are only three we need to study:

Theorem 2.1 *Up to isomorphism, there are just three simply connected Riemann surfaces: Σ, $\mathbb{C}$ and $\mathbb{H}$ (or equivalently $\mathbb{D}$).*

Note that these three Riemann surfaces are mutually non-isomorphic: Σ is compact, whereas $\mathbb{C}$ and $\mathbb{D}$ are not; although $\mathbb{C}$ and $\mathbb{D}$ are homeomorphic, any Riemann surface isomorphism would give a bounded analytic function $\mathbb{C} \to \mathbb{D}$, which must be constant by Liouville's Theorem.

The automorphism groups of these three surfaces have already been described (see Section 1.3), so it is sufficient to study the quotient spaces obtained from their discontinuous subgroups. By an even greater miracle, the first two spaces Σ and $\mathbb{C}$ are rather easily dealt with, and we can concentrate mainly on $\mathbb{H}$, the really interesting and challenging case. In particular, we will see that compact Riemann surfaces of genus 0 and 1 have universal covering surfaces Σ and $\mathbb{C}$, while those of genus $g \geq 2$ have universal covering surface $\mathbb{H}$.

3 Riemann surfaces of genus 0

Here we consider Riemann surfaces X with $\tilde{X} = \Sigma$. Every element of $\operatorname{Aut}\Sigma$ has a fixed point (the equation $(az+b)/(cz+d) = z$ always has a solution!), so the only discontinuous subgroup Γ is the trivial group, with quotient $X \cong \Sigma$. We will see later that the compact quotients obtained from the other simply connected Riemann surfaces $\mathbb{C}$ and $\mathbb{H}$ all have genus $g = 1$ or $g > 1$ respectively. This shows that, up to isomorphism, the only compact Riemann surface of genus 0 is the Riemann sphere Σ.

The rotation group $SO(3, \mathbb{R})$ of S^2 is embedded in $\operatorname{Aut}\Sigma$ as the projective special unitary group $PSU(2, \mathbf{C})$, consisting of the Möbius transformations with $c = -\bar{b}, d = a$ and $|a|^2 + |b|^2 = 1$. This is a compact group, homeomorphic to the quotient of S^3 ($= SU(2, \mathbf{C}) =$ the multiplicative group of unit quaternions) by its antipodal isometry. The fact that S^3 is simply connected means that $SO(3, \mathbb{R})$ has fundamental group C_2, which 'explains' phenomena such as spin in particle physics.

Every finite subgroup of $\operatorname{Aut}\Sigma$ is conjugate to a subgroup of $PSU(2, \mathbf{C})$, so it is isomorphic to C_n, D_n, A_4, S_4 or A_5; the last three are the rotation groups of the regular solids. Lifting these to $SU(2, \mathbb{C})$ we get their double covers, the binary polyhedral groups.

4 Riemann surfaces of genus 1

4.1 Tori

Next we consider Riemann surfaces X with $\tilde{X} = \mathbb{C}$. A non-identity automorphism $z \mapsto az + b$ of $\mathbb{C}$ has a fixed point if and only if $a \neq 1$, so the discontinuous subgroups Γ of $\operatorname{Aut}\mathbb{C}$ consist of translations $z \mapsto z + b$. Apart from the trivial group, they are either infinite cyclic, generated by a single translation $z \mapsto z + b$, or isomorphic to $\mathbf{Z}^2$, generated by two translations $z \mapsto z + b_1$ and $z \mapsto z + b_2$, where b_1 and b_2 are linearly independent over $\mathbb{R}$.

If Γ is an infinite cyclic translation group, then $\mathbb{C}/\Gamma$ is not compact. In fact, if Γ is generated by $z \mapsto z + b$ then the Γ-invariant analytic function $z \mapsto \exp(2\pi i z/b)$ induces an isomorphism $\mathbb{C}/\Gamma \to \mathbb{C} \setminus \{0\}$ of Riemann surfaces. Note by Example 2 in Section 2.1 that this surface X has fundamental group $\pi_1 X \cong \mathbb{Z} \cong \Gamma$.

A group Γ of the second type is called a *lattice*, which I will denote by Λ, and the generators $z \mapsto z + b_1$ and $z \mapsto z + b_2$ form a *basis*; by identifying each translation $z \mapsto z + b$ with the element $b \in \mathbb{C}$, we can regard Λ as an additive subgroup of $\mathbb{C}$.

If a group Γ acts on a topological space T by homeomorphisms, then a subset $F \subseteq T$ is a *fundamental region* for Γ if

- F is connected;
- every element of T is equivalent under Γ to an element of F;
- if two elements of F are equivalent under Γ they are both on the boundary of F.

Thus, apart from some possible duplication on the boundary, F is a connected set of representatives for the orbits of Γ on T, so by identifying equivalent boundary points of F we get the quotient space T/Γ as F/Γ. In the case of a lattice Λ acting on $\mathbb{C}$, we can take the parallelogram with vertices $0, b_1, b_2$ and $b_1 + b_2$ as a fundamental region. Identifying opposite sides (which are equivalent under Λ) we see that $\mathbb{C}/\Lambda$ is topologically a torus.

The images of a fundamental region F under Γ tessellate T, overlapping only at their common boundaries; in the case of a lattice, we get a tessellation of $\mathbb{C}$ by parallelograms.

4.2 Isomorphisms of tori

One can show two tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic (as Riemann surfaces) if and only if the lattices Λ and Λ' are conjugate in $\operatorname{Aut}\mathbb{C} = AGL(1,\mathbb{C})$, or equivalently Λ and Λ' are similar, that is, $\Lambda' = a\Lambda$ for some $a \in \mathbf{C} \setminus \{0\}$. The *modulus* of a basis b_1, b_2 for Λ is $\tau = b_1/b_2$, where we choose the numbering so that $\operatorname{Im}\tau > 0$, that is, $\tau \in \mathbb{H}$. Similarity of lattices leaves τ unchanged. The other bases for Λ have the form

$$b_1' = ab_1 + bb_2, \quad b_2' = cb_1 + db_2 \quad (a, b, c, d \in \mathbb{Z},\ ad - bc = 1)$$

(we need $ad - bc = 1$ rather than -1 to ensure that $b_1'/b_2' \in \mathbb{H}$), giving a modulus

$$\tau' = \frac{ab_1 + bb_2}{cb_1 + db_2} = \frac{a\tau + b}{c\tau + d}.$$

Thus moduli $\tau, \tau' \in \mathbb{H}$ correspond to similar lattices, and hence to isomorphic tori, if and only if they are equivalent under the action of the *modular group*

$$PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm I\}$$

consisting of the Möbius transformations

$$f : z \mapsto \frac{az+b}{cz+d} \quad (a,b,c,d \in \mathbb{Z},\ ad-bc=1).$$

As a fundamental region for the action of this group on $\mathbb{H}$ we can take

$$F = \{z \in \mathbb{H} \mid |z| \geq 1, |\mathrm{Re}\, z| \leq 1/2\}.$$

The element $Z : z \mapsto z+1$ pairs the sides $\mathrm{Re}\, z = -1/2$ and $\mathrm{Re}\, z = 1/2$ of F, and the element $X : z \mapsto -1/z$, fixing i, pairs the two halves of the side $|z| = 1$. Their product $Y = XZ : z \mapsto -1/(z+1)$, fixing $\omega = e^{2\pi i/3}$, has order 3, and one can show that

$$PSL(2,\mathbb{Z}) = \langle X, Y \mid X^2 = Y^3 = 1\rangle \cong C_2 * C_3,$$

where $*$ denotes a free product.

To summarise: the isomorphism classes of tori correspond to the orbits of $PSL(2,\mathbb{Z})$ on $\mathbb{H}$, or equivalently to the elements of F with the above boundary identifications; the parameter τ indicates that the uniformising lattices are similar to that with basis $\{\tau, 1\}$.

4.3 Automorphism groups of tori

One can show that the automorphisms of a torus $X = \mathbb{C}/\Lambda$ are induced by $N(\Lambda)$, the normaliser of Λ in $\mathrm{Aut}\,\mathbb{C} = AGL(1,\mathbb{C})$, acting on $\mathbb{C}$. The kernel of its action on X is Λ, so

$$\mathrm{Aut}\, X \cong N(\Lambda)/\Lambda.$$

Now $N(\Lambda)$ consists of the affine transformations $z \mapsto az+b$ with $a, b \in \mathbb{C}$ and $a\Lambda = \Lambda$. For most lattices Λ, the only such values of a are ± 1, in which case $\mathrm{Aut}\, X$ is a semidirect product of $\mathbb{C}/\Lambda \cong S^1 \times S^1$ by C_2, with the generator of C_2 inverting $\mathbb{C}/\Lambda$ by conjugation. The exceptions arise when Λ is a square lattice, with $\tau = i$, or a triangular lattice, with $\tau = \omega$ (or equivalently $\omega + 1 = e^{2\pi i/6}$), in which cases the complement is C_4 or C_6 respectively. Thus, like Σ, each torus has an uncountable automorphism group; in the case of Σ it is a simple group, but here they are cyclic extensions of abelian groups, and thus solvable.

4.4 Elliptic curves

A major theorem, essentially due to Riemann, states that compact Riemann surfaces are equivalent to complex algebraic curves (defined by polynomial equations with coefficients in $\mathbb{C}$), and vice versa. For instance, every torus $\mathbb{C}/\Lambda$ can be regarded as an *elliptic curve* E, defined by an equation

$$y^2 = p(x)$$

where p is a cubic polynomial in $\mathbb{C}[x]$ with distinct roots.

To see this, given any lattice Λ we define the *Weierstrass function*

$$\wp(z) = \wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda}{}' \Big(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \Big),$$

where $\sum'$ denotes summation over all non-zero $\lambda \in \Lambda$. (Unfortunately, the more natural-looking $\sum (z-\lambda)^{-2}$ doesn't converge.) One can show that $\wp(z)$ is an elliptic function with respect to Λ, i.e. it is meromorphic (with poles of order 2 at the lattice points), and doubly periodic (i.e. $\wp(z+\lambda) = \wp(z)$ for all $z \in \mathbb{C}$ and $\lambda \in \Lambda$). Its derivative

$$\wp'(z) = -2 \sum_{\lambda\in\Lambda}{}' \frac{1}{(z-\lambda)^3}$$

is also an elliptic function. By comparing their Laurent series near 0 one can show that these two functions satisfy a differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where

$$g_2 = g_2(\Lambda) = 60 \sum_{\lambda\in\Lambda}{}' \frac{1}{\lambda^4} \quad \text{and} \quad g_3 = g_3(\Lambda) = 140 \sum_{\lambda\in\Lambda}{}' \frac{1}{\lambda^6}.$$

Parametrising E by putting $x = \wp(z)$ and $y = \wp'(z)$, we identify it with $X = \mathbb{C}/\Lambda$; points $z \in \Lambda$ correspond to the 'point at infinity' (∞, ∞) on E, which is a sign that to be precise we should really regard E as a projective curve, rather than an affine curve.

(This process is analogous to parametrising the algebraic curve $y^2 = 1 - x^2$ by using the simply periodic functions $x = \sin z$ and $y = \sin' z = \cos z$.)

One can show that the discriminant $g_2^3 - 27g_3^2$ of the cubic polynomial on the right-hand side of the differential equation is non-zero, so it has distinct roots. Conversely, given any cubic polynomial with distinct roots, one can use an affine transformation of the variables to put it into Weierstrass normal form

$$p(x) = 4x^3 - c_2x - c_3,$$

and one can then show that there is a lattice Λ for which $g_2 = c_2$ and $g_3 = c_3$. It follows that the elliptic curve $y^2 = p(x)$ corresponds to the torus $X = \mathbb{C}/\Lambda$.

Remarks. 1. A more elementary way of seeing that an elliptic curve has genus 1 is to construct the Riemann surface of the 2-valued function $y = \sqrt{p(x)}$. If $p(x)$ has distinct roots a, b and c in $\mathbb{C}$, then we take two copies of the Riemann sphere, one for each branch of the function, cut them between a and b and between c and ∞, and join them across the cuts to produce a surface of genus 1.

2. Several of the preceding arguments depend on the properties of a certain analytic function $J : \mathbb{H} \to \mathbb{C}$, defined by

$$J(\tau) = \frac{g_2^3}{g_2^3 - 27g_3^2}$$

where $g_i = g_i(\Lambda)$ for a lattice Λ with modulus τ. This is a *modular function*, i.e. it is invariant under the action of the modular group $PSL(2,\mathbb{Z})$ on $\mathbb{H}$. In particular it is periodic with respect to $\mathbb{Z}$, so it has a Fourier series expansion, as a Laurent series in $q = e^{2\pi i\tau}$. John Mackay observed that some of the coefficients in this expansion are closely associated with the degrees of the irreducible representations of the Monster simple group. This was the starting point of the theory known as 'monstrous moonshine'.

3. Every elliptic curve $y^2 = p(x)$ has an automorphism $(x, y) \mapsto (x, -y)$ of order 2, corresponding to the automorphism $z \mapsto -z$ possessed by every torus. In special cases, one can see automorphisms of other orders: for instance $y^2 = 4x^3 - 1$ has obvious automorphisms of order 2 and 3, with product of order 6; this elliptic curve corresponds to the triangular lattice, which also has such automorphisms.

4. Elliptic curves, being identified with tori $\mathbb{C}/\Lambda$, have an abelian group structure inherited from $\mathbb{C}$. Whereas it is easy to perform group operations using the complex parameter z, it is much harder to do so using the original variables x and y. For this reason, elliptic curves (over finite fields) are currently used in some of the most powerful cryptographic systems.

5. Elliptic curves play a central role in several other areas of mathematics: for instance Andrew Wiles proved Fermat's Last Theorem by proving part of the related Taniyama-Shimura Conjecture (subsequently the Modularity Theorem of Breuil, Conrad, Diamond and Taylor), which establishes a connection between elliptic curves defined over $\mathbb{Q}$ and modular forms.

5 Groups acting on $\mathbb{H}$

Here we develop the ideas needed to consider Riemann surfaces X with universal covering surface $\tilde{X} = \mathbb{H}$, i.e. those uniformised by groups acting discontinuously on $\mathbb{H}$. We shall see that the compact surfaces of this type all have genus $g \geq 2$.

5.1 Automorphisms of $\mathbb{H}$

It is useful to divide the non-identity elements $f \in \mathrm{Aut}\,\mathbb{H} = PSL(2,\mathbb{R})$ into three classes, according to their fixed points. Let

$$f(z) = \frac{az+b}{cz+d} \quad (a, b, c, d \in \mathbb{R},\ ad - bc = 1).$$

Solving $f(z) = z$ gives a quadratic equation $cz^2 + (d-a)z - b = 0$, with real coefficients, and discriminant $D = (d-a)^2 + 4bc = (a+d)^2 - 4$. There are three possibilities:

- If $D < 0$, i.e. $|a+d| < 2$, there are two complex conjugate roots, giving a single fixed point in $\mathbb{H}$. We call f an *elliptic* element. Example: $z \mapsto -1/z$, fixing $i \in \mathbb{H}$.

- If $D = 0$, i.e. $|a+d| = 2$, there is a single root in $\partial\mathbb{H} = \hat{\mathbb{R}}$ and so no fixed points in $\mathbb{H}$. We call f a *parabolic* element. Example: $z \mapsto z + \lambda$ ($\lambda \in \mathbb{R}$), fixing $\infty \in \partial\mathbb{H}$.

- If $D > 0$, i.e. $|a+d| > 2$, there are two distinct real roots, and so no fixed points in $\mathbb{H}$. We call f a *hyperbolic* element. Example: $z \mapsto \lambda z$ $(1 \neq \lambda > 0)$, with $a = \sqrt{\lambda} = 1/d$, fixing $0, \infty \in \partial\mathbb{H}$.

Note that the matrices $\pm A \in SL(2,\mathbb{R})$ reprsenting f have trace $\pm(a+d)$, so we say that f has *trace* $\mathrm{tr}(f) = \pm(a+d)$, or that $\mathrm{tr}^2(f) = (a+d)^2$.

One can use this classification to show that two non-identity elements of $PSL(2,\mathbb{R})$ commute if and only if they have the same fixed points in $\mathbb{H} \cup \partial\mathbb{H}$.

5.2 Fuchsian groups

Riemann surfaces X with $\tilde{X} = \mathbb{H}$ are uniformised by discontinuous subgroups K of $PSL(2,\mathbb{R})$ (the analogues of the lattices in $\mathbb{C}$). These must act without fixed points on $\mathbb{H}$. As in the case of tori, the automorphisms of X are induced by the normaliser

$$N(K) = \{g \in PSL(2,\mathbb{R}) \mid g^{-1}Kg = K\}$$

of K in $PSL(2,\mathbb{R})$:

Theorem 5.1 *If a Riemann surface X is uniformised by a subgroup $K \leq PSL(2,\mathbb{R})$, then* $\mathrm{Aut}\,X \cong N(K)/K$.

Elements of $N(K)$ may have fixed points in $\mathbb{H}$, in which case this group does not act discontinuously; we therefore need a weaker concept than discontinuity to cover such groups. A subgroup Γ of $PSL(2,\mathbb{R})$ acts *properly discontinuously* on $\mathbb{H}$ if each point $p \in \mathbb{H}$ has an open neighbourhood U such that if $U \cap g(U) \neq \emptyset$ for some $g \in \Gamma$ then $g(p) = p$. Thus fixed points are allowed, but orbits cannot accumulate. Discontinuous implies properly discontinuous, but not conversely (consider the group generated by $z \mapsto -1/z$, fixing i).

Theorem 5.2 *A subgroup Γ of $PSL(2,\mathbf{R})$ acts properly discontinuously on $\mathbb{H}$ if and only if it is discrete.*

Here the topology on $PSL(2,\mathbb{R})$ is that inherited from $SL(2,\mathbb{R})$, regarded as a subset of $\mathbb{R}^4$. Groups Γ satisfying these equivalent conditions are called *Fuchsian groups*.

Example. The modular group $\Gamma = PSL(2,\mathbb{Z})$ is a discrete subgroup of $PSL(2,\mathbb{R})$, since $\mathbb{Z}$ is a discrete subring of $\mathbb{R}$. It acts properly discontinously, but not discontinuously since it contains $z \mapsto -1/z$.

Theorem 5.3 *If K is a non-abelian Fuchsian group, then its normaliser $N(K)$ is also a Fuchsian group.*

Non-identity commuting elements of $PSL(2,\mathbb{R})$ have the same fixed points, so an abelian subgroup of $PSL(2,\mathbb{R})$ must fix a point in $\mathbb{H} \cup \partial\mathbb{H}$; considering the point-stabiliaers shows that if it acts properly discontinuously on $\mathbb{H}$ then it is cyclic. In particular, it cannot be isomorphic to the fundamental group $\mathbb{Z}^2$ of a compact surface of genus 1. We therefore have:

Theorem 5.4 *If X is a compact Riemann surface which is uniformised by a subgroup $K \leq PSL(2, \mathbb{R})$, then X has genus $g \geq 2$ and $N(K)$ is a Fuchsian group.*

This, together with Theorem 5.1, means that we can study the automorphism groups of compact Riemann surfaces of genus $g \geq 2$ by considering Fuchsian groups Γ and their quotients Γ/K by discontinuous normal subgroups K.

5.3 Hyperbolic geometry

In constructing examples of Fuchsian groups, it is useful to regard $\mathbb{H}$ as a model of the *hyperbolic plane*, with metric dz/y for $z = x + iy$, so that the length of a curve $\gamma : I = [0, 1] \to \mathbb{H}, t \mapsto z = \gamma(t)$ is

$$\int_0^1 \frac{1}{y}\left|\frac{dz}{dt}\right| dt.$$

The geodesics are the intersections of $\mathbb{H}$ with Euclidean semicircles and straight lines perpendicular to $\mathbb{R}$. Then we have

Theorem 5.5 *$PSL(2, \mathbb{R})$ is the group of orientation-preserving isometries of $\mathbb{H}$.*

(Composing its elements with the reflection $z \mapsto -\overline{z}$ gives the orientation-reversing isometries; the whole isometry group is isomorphic to $PGL(2, \mathbb{R})$, but note that the action of elements of $PGL(2, \mathbb{R}) \setminus PSL(2, \mathbb{R})$ is 'twisted' by complex conjugation.)

Lemma 5.6 *In a Fuchsian group Γ, every elliptic element has finite order.*

Proof. An elliptic element $f \in \Gamma$ has a fixed point $p \in \mathbb{H}$, and is a hyperbolic rotation by an angle θ around p. If f has infinite order then θ is an irrational multiple of π, so the orbits of $\langle f \rangle$ are dense subsets of hyperbolic circles centred at p, and hence Γ does not act properly discontinuously. Thus f must have finite order. $\square$

Using the disc model $\mathbb{D}$, it is easy to see that there are elliptic transformations of all orders. However, parabolic and hyperbolic transformations all have infinite order.

We define the *measure* $\mu(\Gamma)$ of a Fuchsian group Γ to be the hyperbolic area of $\mathbb{H}/\Gamma$, or equivalently of a fundamental region for Γ. If Δ is a subgroup of Γ, then a fundamental region for Δ is a union of $|\Gamma : \Delta|$ copies of a fundamental region for Γ, one for each coset. These meet only at their boundaries, so we have the very useful Riemann-Hurwitz Formula:

$$\mu(\Delta) = |\Gamma : \Delta|\mu(\Gamma).$$

6 Constructing Fuchsian groups

We are now ready to construct some specific examples of Fuchsian groups, and to consider their presentations.

6.1 Triangle groups

Let T be a hyperbolic triangle in $\mathbb{H}$, with internal angle $\pi/l, \pi/m, \pi/n$ at vertices v_1, v_2, v_3 for integers $l, m, n \geq 2$. Since the sum of the internal angles is less than π, we must have

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1.$$

Conversely, if this inequality is satisfied then there is, up to isometries, a unique such triangle. The reflections r_i of $\mathbb{H}$ in the sides of T opposite v_i generate a group $\Delta[l, m, n]$ of isometries of $\mathbb{H}$, called an *extended triangle group*; the triangle T is a fundamental region for this group, so the images of T tessellate $\mathbb{H}$. This tessellation can be used to show that the group acts properly discontinuously on $\mathbb{H}$, and also, using the fact that $\mathbb{H}$ is simply connected, that it has a presentation

$$\Delta[l, m, n] = \langle r_1, r_2, r_3 \mid (r_2 r_3)^l = (r_3 r_1)^m = (r_1 r_2)^n = 1 \rangle.$$

One can see these are defining relations by considering the dual tessellation of $\mathbb{H}$, which embeds a Cayley graph for this group: any word $w = 1$ represents a closed path from 1 to 1 in the graph, and since $\mathbf{H}$ is simply connected this can be reduced to a constant path by successively using the relations to reduce w. Similar remarks apply to triangle groups acting on $\mathbb{C}$ or Σ, with $l^{-1} + m^{-1} + n^{-1} \geq 1$,. More generally, Poincaré described a method of obtaining a presentation of a properly discontinuous group acting on simply connected surface by considering how a fundamental region meets its neighbours (see Zieschang, Vogt and Coldewey, *Surfaces and Planar Discontinuous Groups*).

The orientation-preserving subgroup of index 2 in $\Delta[l, m, n]$ is the *triangle group* $\Delta = \Delta(l, m, n)$. It acts properly discontinuously as a group of automorphisms of $\mathbb{H}$, so it is a Fuchsian group. It consists of the elements of even length in the generators r_i, so it is generated by elliptic elements $x_1 = r_2 r_3$, $x_2 = r_3 r_1$ and $x_3 = r_1 r_2$ which rotate $\mathbb{H}$ around v_1, v_2 and v_3 through angles $2\pi/l, 2\pi/m, 2\pi/n$. It has a presentation

$$\Delta(l, m, n) = \langle x_1, x_2, x_3 \mid x_1^l = x_2^m = x_3^n = x_1 x_2 x_3 = 1 \rangle.$$

The triangle T and any of the three adjacent triangles $r_i(T)$ form a fundamental region F for Δ. By the Gauss-Bonnet Theorem, T has hyperbolic area

$$\mu(T) = \pi\Big(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\Big),$$

so

$$\mu(\Delta) = \mu(F) = 2\pi\Big(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\Big).$$

One can generalise this construction by allowing T to have *ideal vertices* with internal angle 0 on $\partial\mathbb{H}$. In this case the corresponding generator x_i of Δ is parabolic, of order l, m or $n = \infty$, the relation $x_i^\infty = 1$ is omitted, and in the area formula we regard $1/\infty$ as 0.

Example. Let T have vertices i, $\omega = e^{2\pi i/3}$ and $c = \infty$, with internal abgles $\pi/2$, $\pi/3$ and 0. The resulting extended triangle group $\Delta[2, 3, \infty]$ is the extended modular group $PGL(2, \mathbb{Z})$, and the orientation-preserving triangle group $\Delta(2, 3, \infty)$ is the modular group $\Gamma = PSL(2, \mathbb{Z})$, generated by rotations x_1 of order 2 about i, x_2 of order 3 about b, and a parabolic element x_3 fixing ∞. The ideal triangle with vertices $\omega, \omega + 1$ and ∞ is a fundamental region F for Γ. It is not compact, but it has finite area:

$$\mu(\Gamma) = \mu(F) = 2\pi\Big(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{\infty}\Big) = \frac{\pi}{3}.$$

There is also a natural generalisation to *polygonal groups*, where the triangle T is replaced with a hyperbolic polygon having internal angles π/m_i for integers $m_1 \geq 2$ or $m_i = \infty$. In this case the presentation is

$$\Delta(m_1, \ldots, m_r) = \langle x_1, \ldots, x_r \mid x_i^{m_1} = \ldots = x_r^{m_r} = x_1 \ldots x_r = 1\rangle.$$

6.2 Surface groups

Just as each Riemann surface X of genus 1 is uniformised by a lattice, with a Euclidean parallelogram as a fundamental region, its sides paired by generators, there is a similar situation when X has genus $g \geq 2$, except that now we obtain a hyperbolic polygon, with $4g$ sides, as a fundamental region.

A subgroup K of $PSL(2, \mathbb{R})$ is *cocompact* if it has a compact quotient space $X = \mathbb{H}/K$. If a discontinuous subgroup K of $PSL(2, \mathbb{R})$ is cocompact, then it has a fundamental region F consisting of a hyperbolic polygon with $4g$ sides $A_1', B_1', A_1, B_1, \ldots A_g', B_g', A_g, B_g$ in that cyclic order, with hyperbolic elements a_i, b_i pairing A_i' with A_i, and B_i' with B_i. Then K has a presentation

$$K = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1\rangle,$$

where $[a_i, b_i]$ is the commutator $a_i^{-1}b_i^{-1}a_ib_i$. Thus $K \cong \pi_1 X$, so we call K a *surface group* (see Example 4 of Section 2,1). By decomposing F into triangles, and adding their areas, one can show that

$$\mu(K) = \mu(F) = 4\pi(g - 1).$$

6.3 Presentations of Fuchsian groups

A Fuchsian group Γ is *cofinite* if its quotient space $\mathbb{H}/\Gamma$ has finite area. In such a case, and in particular if it is cocompact, then Γ has a fundamental region with finitely many sides, and the corresponding side-pairing elements generate the group. By choosing a suitable fundamental region, one can use Poincaré's method to find a *standard presentation* for Γ, generalising those given earlier for triangle groups and surface groups. There are generators

$$a_1, b_1, \dots, a_g, b_g \text{ (hyperbolic)}, \quad x_1, \dots, x_r \text{ (elliptic)}, \quad y_1, \dots, y_s \text{ (parabolic)},$$

and defining relations

$$x_1^{m_1} = \cdots = x_r^{m_r} = [a_1, b_1] \dots [a_g, b_g] x_1 \dots x_r y_1 \dots y_s = 1,$$

with integers $m_i \geq 2$ (their order is irrelevant). We summarise this by saying that Γ has *signature* $(g; m_1, \dots, m_r; s)$. Thus a cocompact triangle group $\Delta(l, m, n)$ has signature $(0; l, m, n; 0)$, while $PSL(2, \mathbb{Z})$ has signature $(0; 2, 3; 1)$, and a surface group of genus g has signature $(g; -; 0)$. Here the subgroups $\langle x_i \rangle$, each of order m_i, are representatives of the conjugacy classes of maximal elliptic subgroups of Γ, while the subgroups $\langle y_i \rangle$, each of infinite order, are representatives of the conjugacy classes of maximal parabolic subgroups. The quotient space $\mathbb{H}/\Gamma$ is a surface of genus g with s points removed, so it is compact if and only if $s = 0$, that is, Γ contains no parabolic elements.

Using the Gauss-Bonnet Theorem, one can prove the following generalisation of several earlier formulae:

Theorem 6.1 *If Γ is a Fuchsian group with signature $(g; m_1, \dots, m_r; s)$, then*

$$\mu(\Gamma) = 2\pi\Big(2g - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big) + s\Big).$$

For such a group Γ to exist, it is clearly necessary that the right-hand side should be positive. This is also sufficient:

Theorem 6.2 *There exists a Fuchsian group Γ with signature $(g; m_1, \dots, m_r; s)$ if and only if*

$$2g - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big) + s > 0.$$

This is proved by constructing a fundamental region with appropriate side-pairing elements, and letting Γ be the group they generate.

Theorem 6.3 *If X is a compact Riemann surface of genus $g \geq 2$ then* $\mathrm{Aut}\, X$ *is finite.*

Proof. Since $g \geq 2$, the Uniformisation Theorem tells us that $X \cong \mathbb{H}/K$ for some surface group K. Now $\mathrm{Aut}\, X \cong N(K)/K$ by Theorem 5.1, and $N(K)$ is a Fuchsian group by Theorem 5.4. Since $N(K)$ contains K, and $\mu(K)$ is finite, $\mu(N(K))$ is also finite, and is non-zero by Theorem 6.2. Thus

$$|\mathrm{Aut}\, X| = |N(K) : K| = \frac{\mu(K)}{\mu(N(K))}$$

by the Riemann-Hurwitz Formula, and this is finite. □

Schwarz's original proof used analytic methods involving Weierstrass points on X. We will give an upper bound for $|\mathrm{Aut}\, X|$, due to Hurwitz, in Section 7.4.

7 Riemann surfaces of genus greater than 1

Motivated by Theorem 6.3, we now look for interesting finite groups acting on compact Riemann surfaces of genus $g \geq 2$. In a certain sense (which can be made precise using Teichmüller theory), such surfaces are exceptional: most Riemann surfaces of genus $g = 2$ or $g > 2$ have automorphisms groups of order 2 or 1 respectively. On the other hand, Accola and Maclachlan independently proved that for each $g \geq 2$ there is a Riemann surface of genus g with at least $8(g+1)$ automorphisms.

7.1 Smooth epimorphisms

Suppose that a finite group G acts as a group of automorphisms of a compact Riemann surface X of genus $g \geq 2$. Then X is uniformisied by a surface subgroup K of $PSL(2, \mathbb{R})$, and there is a Fuchsian group $\Gamma \leq N(K)$ such that $\Gamma/K \cong G$. Being cocompact, K has no parabolic elements, and hence neither has Γ, for otherwise a suitable power of one would be a parabolic element of K. Thus Γ has signature $(g'; m_1, \ldots, m_r; 0)$ where g' is the genus of $\mathbb{H}/\Gamma$. Similarly, since K contains no elliptic elements, the natural epimorphism $\theta : \Gamma \to G$, with kernel K, must send each elliptic generator x_i to an element of order m_i in G. We call such an epimorphism *smooth*, or a *surface kernel epimorphism*. Conversely, if $\theta : \Gamma \to G$ is a smooth epimorphism, then $K = \ker\theta$ is a normal surface subgroup of Γ, so G acts as a group of automorphisms of the Riemann surface $\mathbb{H}/K$.

7.2 Modular surfaces

For each prome p, the group $G = L_2(p) = PSL(2, \mathbb{Z}_p)$ has generators g_1, g_2, g_3 of orders $2, 3$ and p with $g_1 g_2 g_3 = 1$; these are the images of the generators x_i of orders $2, 3$ and ∞ of $PSL(2, \mathbb{Z}) = \Delta(2, 3, \infty)$ under reduction mod (p). It follows that there is a smooth epimorphism $\theta : \Delta = \Delta(2, 3, p) \to G$. Now Δ is a Fuchsian group provided $p \geq 7$, in which case we obtain a surface group $K = \ker\theta$, uniformising a compact Riemann surface X with $G \leq \operatorname{Aut} X$. (In fact, a theorem of Singerman [*J. London Math. Soc.* 1972] shows that Δ is maximal among Fuchsian groups, so $\Delta = N(K)$ and hence $G = \operatorname{Aut} X$.) Now

$$|\Delta : K| = |G| = \frac{1}{2}p(p^2 - 1),$$

and

$$\mu(\Delta) = 2\pi\Big(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{p}\Big) = \frac{(p-6)\pi}{3p},$$

so the Riemann-Hurwitz Formula gives

$$\mu(K) = \frac{(p^2-1)(p-6)\pi}{6p}.$$

If X has genus g then $\mu(K) = 4\pi(g-1)$, so we have

$$g = \frac{(p+2)(p-3)(p-5)}{24}.$$

These surfaces X are called *modular surfaces*. The first example, with $p = 7$, has genus $g = 3$, with the simple group $L_2(7)$ of order 168 as its automorphism group. It is knon as Klein's quartic curve, since it is given as a projective algebraic curve by the equation

$$x^3y + y^3z + z^3x = 0.$$

(There is a wonderful book, *The Eightfold Way: the Beauty of Klein's Quartic Curve*, edited by Silvio Levy, on the mathematical, historical and aesthetic aspects of this surface.)

It is not essential that g_3 should have prime order here. One can apply the same construction with $G = PSL(2, \mathbb{Z}_n)$ for any integer $n \geq 7$, though now G, which need not be simple, has order

$$\frac{n^3}{2}\prod_{p|n}\Big(1-\frac{1}{p^2}\Big),$$

where p ranges over the distinct primes dividing n.

7.3 Counting smooth epimorphisms

Finding a smooth epimorphism from a triangle group $\Delta(l, m, n)$ onto a finite group G is equivalent to finding a generating triple g_1, g_2, g_3 of orders l, m and n in G, with $g_1g_2g_3 = 1$. One can count solutions of this last equation by using character theory.

If $A_1, \ldots, A_r$ are conjugacy classes in a finite group G, then the number of r-tuples $(g_1, \ldots, g_r) \in A_1 \times \cdots \times A_r$ such that $g_1 \ldots g_r = 1$ in G is given by the expression

$$\frac{|A_1|.\ \ldots\ .|A_r|}{|G|}\sum_{\chi}\frac{\chi(g_1)\ldots\chi(g_r)}{\chi(1)^{r-2}},$$

where $g_i \in A_i$ and χ ranges over the irreducible complex characters of G. (For a proof, see Theorem 7.2.1 in *Topics in Galois Theory* by J-P. Serre; heuristically, one might expect the formula to be $\prod_i |A_i|/|G|$, the number of r-tuples divided by the number of possible values for their product; the extra factor, involving characters, tells us that finite groups do not behave quite as uniformly as we might like, and it also saves us from the possible embarassment of producing a number which is not an integer.) In the particular case $r = 3$ the number of triples is

$$\frac{|A_1|.|A_2|.|A_r|}{|G|}\sum_{\chi}\frac{\chi(g_1)\chi(g_2)\chi(g_r)}{\chi(1)}.$$

In order to have a smooth homomorphism, we choose the conjugacy classes A_i to consist of elements g_i of the same orders as the corresponding elliptic generators x_i of Δ. If one can show that some r-tuple (g_i) generates G (for instance, by showing that no maximal subgroup contains all the elements g_i), then we have an epimorphism $\Delta \to G$. A more sophisticated approach to this, due to P. Hall, uses Möbius inversion in the lattice of subgroups of G; see [Jones, *Quarterly J. Math.* 1995] for some applications.

For any pair of groups Δ and G, two epimorphisms $\Delta \to G$ have the same kernel if and only if they differ by an automorphism of G; since $\mathrm{Aut}\, G$ acts fixed-point-freely on generating sets, it has orbits of length $|\mathrm{Aut}\, G|$ on thse epimorphisms, so one can count kernels K by dividing the number of epimorphisms by $|\mathrm{Aut}\, G|$.

Example 1. Let $G = L_2(7)$ with $\Delta = \Delta(2,3,7)$, as in Section 7.2 with $p = 7$. The character table of G is shown below, with each column headed by the order of the elements and the number of them in the corresponding conjugacy class; the entries α and β denote $(-1 \pm i\sqrt{7})/2$.

	1	2	3	4	7	7
	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_2	3	-1	0	1	α	β
χ_3	3	-1	0	1	β	α
χ_4	6	2	0	0	-1	-1
χ_5	7	-1	1	-1	0	0
χ_6	8	0	-1	0	1	1

Character table of $L_2(7)$

We see that there is one conjugacy class A_1 of 21 involutions g_1, one conjugacy class A_2 of 56 elements g_2 of order 3, and there are two conjugacy classes A_3 each containing 24 elements g_3 of order 7. For either choice of A_3, each irreducible character $\chi \neq \chi_1$ of G vanishes on at least one class A_i, so the summation in the formula is equal to 1, and the number of triples is

$$\frac{21.56.24}{168} = 168.$$

Counting both choices for A_3 we therefore obtain 336 triples. No proper subgroup of G has order divisible by $2, 3$ and 7, so these triples all generate G. Each triple therefore determines a smooth epimorphism $\theta : \Delta \to G, x_i \mapsto g_i$. Since $\mathrm{Aut}\, G = PGL(2,7)$ has order 336, it has a single orbit on these triples, so there is a single kernel K. Thus Δ provides a single Riemann surface X with $\mathrm{Aut}\, X \cong L_2(7)$: this is Klein's quartic curve of genus 3, as found in Section 7.2.

Example 2. In some cases, we get more than one kernel K. If we take the same group $\Delta = \Delta(2,3,7)$ as before, but now with $G = L_2(13)$, a similar calculation using the character table of G (see the *ATLAS*) shows that there are three normal surface groups K with $\Delta/K \cong G$, one for each of the three conjugacy classes A_3 of elements g_3 of order 7 in G, and hence we obtain three non-isomorphic Riemann surfaces X with $\mathrm{Aut}\, X \cong L_2(13)$. By the Riemann-Hurwitz Formula they have genus $g = 14$.

Example 3. Not all triangle groups are maximal. Let $\Delta = \Delta(3,5,5)$, and let $G = A_5 \cong L_2(5)$. In the character table below, α and β denote $(-1 \pm \sqrt{5})/2$

	1	2	3	5	5
	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	β
χ_3	3	-1	0	β	α
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

Character table of A_5

We see that G has one conjugacy class of 20 elements of order 3, and two conjugacy classes of 12 elements of order 5; these are transposed by squaring and by conjugacy in $\mathrm{Aut}\, G = S_5$. The above method shows that there are two normal surface subgroups K_1, K_2 in Δ with $\Delta/K_i \cong G$, uniformising two Riemann surfaces X_i of genus 9. They are distinguished by the elliptic generators of Δ of order 5 being mapped to conjugate elements in Δ/K_1, and to non-conjugate elements in Δ/K_2. However, G is not the full automophism group of either surface: Singerman's 1972 paper shows that Δ is a subgroup of index 2 in $\Gamma = \Delta(2,5,6)$, which is maximal. One can show that Γ normalises each K_i, so $\Gamma = N(K_i)$ for $i = 1, 2$. We find that $\mathrm{Aut}\, X_1 = \Gamma/K_1 \cong S_5$, while $\mathrm{Aut}\, X_2 \cong \Gamma/K_2 \cong A_5 \times C_2$.

7.4 Hurwitz groups and surfaces

A simple but tedious argument based on the formula in Theorem 6.1 shows that each cofinite Fuchsian group Γ has $\mu(\Gamma) \geq \pi/21$, and that the only groups attaining this bound are those with signature $(0; 2, 3, 7; 0)$; these are the triangle groups $\Delta = \Delta(2,3,7)$. The following result is due to Hurwitz:

Theorem 7.1 *If X is a compact Riemann surface of genus $g \geq 2$, then*

$$|\mathrm{Aut}\, X| \leq 84(g-1).$$

Proof. As in the proof of Theorem 6.3 we have

$$|\mathrm{Aut}\, X| = |N(K) : K| = \frac{\mu(K)}{\mu(N(K))}.$$

Now $\mu(K) = 4\pi(g-1)$, and $\mu(N(K)) \geq \pi/21$, so $|\mathrm{Aut}\, S| \leq 84(g-1)$. $\square$

The Riemann surfaces X attaining this bound are known as *Hurwitz surfaces*, and their automorphism groups G are *Hurwitz groups*. The following is now straightforward:

Theorem 7.2 *If G is a finite group, then the following are equivalent:*

1. *G is a Hurwitz group,*

2. *G is a nontrivial quotient of* $\Delta(2,3,7)$*,*

3. G has generators g_1, g_2, g_3 of orders $2, 3$ and 7 with $g_1 g_2 g_3 = 1$.

Notice that Δ is perfect, and hence so is every Hurwitz group. It therefore makes sense to look first among the nonabelian finite simple groups for Hurwitz groups. The easiest to study are the groups $L_2(q)$. We have already seen that $L_2(7)$ and $L_2(13)$ are Hurwitz groups. Macbeath has generalised this:

Theorem 7.3 *If q is a power of a prime p then the group $L_2(q)$ is a Hurwitz group if and only if*

1. $q = 7$, or

2. $q = p \equiv \pm 1$ mod (7), or

3. $q = p^3$ where $p \equiv \pm 2$ or ± 3 mod (7).

In cases (1) and (3) there is a unique Hurwitz surface, but in case (2) there are three.

Conder has shown that the alternating group A_n is a Hurwitz group for all sufficiently large n, and R. A. Wilson has used massive computing power to show that the Monster simple group is a Hurwitz group.

7.5 Fermat curves

Let $\Gamma = \Delta(2, 3, 2n)$ for some integer $n \geq 2$. There is an obvious epimorphism $\Gamma \to \Delta(2, 3, 2) \cong S_3$, and the kernel Δ is a triangle group $\Delta(n, n, n)$: one can see this by barycentrically subdividing an equilateral triangle with internal angles π/n into six triangles with internal angles $\pi/2, \pi/3$ and π/n, so that these are fundamental regions for the corrresponding extended triangle groups. Now let K be the commutator subgroup Δ' of Δ; this is a characteristic subgroup of Δ, and Δ is normal in Γ, so K is normal in Γ. Now $\Delta^{\mathrm{ab}} = \Delta/\Delta' \cong C_n \times C_n$, and the three elliptic generators of order n in Δ are mapped to elements of order n in this group, so K is a surface group, uniformising a Riemann surface X. If $n \geq 4$ then Γ is a Fuchsian group, and Singerman has shown that it is a maximal Fuchsian group, so $\Gamma = N(K)$. Thus $G := \mathrm{Aut}\, X \cong \Gamma/K$, an extension of $\Delta/K \cong C_n \times C_n$ by $\Gamma/\Delta \cong S_3$.

We have $\mu(\Delta) = 2\pi(1 - \frac{3}{n})$, and $|\Delta : K| = n^2$, so $\mu(K) = 2\pi n^2(1 - \frac{3}{n}) = 2\pi n(n-3)$. If X has genus g then $\mu(K) = 4\pi(g-1)$, so we deduce that

$$g = \frac{(n-1)(n-2)}{2}.$$

We will show that X is the Fermat curve $X_n \subset \mathbb{P}^1(\mathbb{C})$ of exponent n, given by

$$x^n + y^n + z^n = 0.$$

The transformations a and b which multiply x and y respectively by $\zeta = e^{2\pi i/n}$ generate a subgroup $N \cong C_n \times C_n$ of $\mathrm{Aut}\, X_n$. There is also a subgroup $S \cong S_3$, permuting the variables, and one easily checks that the subgroup $G_n := \langle N, S\rangle$ of $\mathrm{Aut}\,X_n$ is a semidirect product of N by S. .We can choose generators u and v of S, of orders 2 and 3, so that u transposes a and b, while v send a to b and b to $a^{-1}b^{-1}$. Then the elements u, bv and $(ubv)^{-1} = abv^2u$ generate G_n, have product 1, and have orders $2, 3$ and $2n$, so G^* is a smooth quotient Γ/K_n of $\Gamma = \Delta(2, 3, 2n)$ for some surface group K_n uniformising X_n. One easily checks that Γ has a unique normal subgroup with quotient S_3, namely Δ, and that Δ has a unique normal subgroup with quotient $C_n \times C_n$, namely K, so $K = K_n$ and hence $X = X_n$. This also shows that $G = G_n$, so G is a split extension of $C_n \times C_n$ by S_3.

8 Action on homology

If a Riemann surface is uniformised by a surface group K, then $K \cong \pi_1 X$, so the first integer homology group $H_1 := H_1(X; \mathbb{Z})$ is isomorphic to $(\pi_1 X)^{\mathrm{ab}} \cong K^{\mathrm{ab}} = K/K'$.

There is a natural action of $G := \mathrm{Aut}\,X$ on H_1, induced by its action on X. We can also see this action group-theoretically. The normaliser $\Gamma = N(K)$ of K acts by conjugation on K, leaving K' invariant, so it has an induced action on K^{ab}. By definition of K', K is in the kernel of this action, so there is an induced action of Γ/K on K^{ab}, and hence of G on H_1. This is the natural action on homology.

If R is any commutative ring, then $H_1(X, R) \cong H_1 \otimes_{\mathbb{Z}} R$ as a G-module, with G acting trivially on R. Taking $R = \mathbb{C}$ we can apply ordinary representation theory to this action of G. Alternatively, taking $R = \mathbb{Z}_p$, i.e. reducing coefficients mod (p) and considering the action of G on $K/K'K^p$, we can apply modular representation theory. This technique is particularly useful in studying regular abelian coverings of Riemann surfaces.

Example. Let $G = L_2(7)$, acting on Klein's quartic X of genus $g = 3$. Then H_1 has dimension $2g = 6$, and over $\mathbb{C}$ the representation of G splits as the sum of its two irreducible representations ρ_2 and ρ_3 of degree 3. These are complex conjugates of each other, and correspond to the actions of G on the spaces of holomorphic and anti-holomorphic differentials on X.

If we reduce mod (2), we find that $H_1(X; \mathbb{Z}_2)$ also affords the direct sum of two absolutely irreducible representations of degree 3. One is the natural representation of G as $L_3(2) = GL(3, 2)$ on a vector space $V = C_2^3$, and the other is the dual representation on V^*. These two representations are transposed by the outer automorphism of G, induced by conjugation in $PGL(2, 7)$. This shows that, between K and $K'K^2$, there are two normal subgroups K_1 and K_2 of Γ, with $|K : K_i| = |K_i : K'K^2| = 8$, giving a pair of Hurwitz surfaces X_1 and X_2 which are 8-sheeted coverings of X. These subgroups K_i are conjugate in the extended triangle group $\Delta[2, 3, 7]$, so X_1 and X_2 are complex conjugates of each other; in particular, they are not defined over a real field.

If we reduce mod (3), however, we find that $H_1(X; \mathbb{Z}_3)$ is an irreducible G-module, so we obtain a single Hurwitz surface, a 3^6-sheeted covering of X. There is a similar phenomenon if we reduce mod (5), whereas $H_1(X; \mathbb{Z}_7)$ is a reducible but indecomposable G-module: there is a single G-invariant proper submodule, of dimension 3, so we obtain two Hurwitz surfaces, 7^3- and 7^6-sheeted coverings of X.

9 Riemann surfaces and algebraic number fields

Compact Riemann surfaces are equivalent to algebraic curves defined over $\mathbb{C}$, i.e. defined by polynomial equations with complex coefficients. Which compact Riemann surfaces X are (or rather can be) defined over the subfield $\overline{\mathbb{Q}}$ of algebraic numbers (the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$? Belyĭ's Theorem (1979) states these these are the Riemann surfaces uniformised by subgroups of finite index in triangle groups. For instance, all the examples of compact Riemann surfaces discussed in Section 7 satisfy this condition. Now a triangle group induces a tessellation of the space $\mathbb{H}$ or $\mathbb{C}$ it acts on, and this induces a tessellation of X. For instance, Klein's quartic curve, which is defined over $\mathbb{Q}$ and hence certainly over $\overline{\mathbb{Q}}$, is tessellated by 24 7-gons, or dually by 56 triangles. These tessellations, or various maps on surfaces equivalent to them, were called *dessins d'enfants* (children's drawing) by Grothendieck, because that is what many of them resemble. He pointed out that the *absolute Galois group* $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ acts on these, by acting on the coefficients of the defining polynomials, and that this action is faithful. This is surprising, because $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ is a very complicated profinite group (the projective limit of the finite Galois groups of all the algebraic number fields), and yet one can (at least in theory) see all of it through its action on these rather simple combinatorial objects.

A more recent development has been the use of these quotients of triangle groups by algebraic geometers, such as Beauville, Catanese and others, to construct complex surfaces (i.e. 2-dimensional varieties over $\mathbb{C}$, so 4-dimensional as real manifolds), with interesting geometric properties, such as rigidity (absence of deformations). One takes two compact Riemann surfaces (i.e. algebraic curves) X_i $(i = 1, 2)$ of genus $g_i \geq 2$, uniformised by normal subgroups K_i of triangle groups Δ_i such that $\Delta_1/K_1 \cong \Delta_2/K_2$, and such that this finite group $G \cong \Delta_i/K_i$ acts fixed-point-freely on $X_1 \times X_2$, i.e. no non-identity element has a fixed point on both Riemann surfaces. Then $(X_1 \times X_2)/G$ is a complex surface, called a *Beauville surface*. There is a conjecture that every non-abelian finite simple group except A_5 has such a pair of actions, and several teams involving Lubotzky, Magaard and others seem to be close to a proof of it.